J. Fluid Mech. (1975), *vol.* 70, *part* 2, *pp.* 251–255
Printed in Great Britain

Statistical properties of the kinematics and dynamics of a random gravity-wave field

By C. C. TUNG
Department of Civil Engineering, North Carolina State University, Raleigh

(Received 17 September 1974)

The probability density function and the first three statistical moments of the velocity, acceleration and dynamic pressure are obtained for a Gaussian, stationary, homogeneous, random gravity-wave field in deep water, using infinitesimal wave solutions. It is shown that the velocity, acceleration and pressure are non-Gaussian. While the horizontal accelerations and vertical velocity component are of zero mean and unskewed, the dynamic pressure, vertical acceleration and horizontal velocity components are skewed and have non-zero mean.

1. Introduction

Fluid motion in a random wave field has been of interest to oceanographers and engineers for many years. This is particularly true for coastal engineers, who are concerned with the forces on marine structures and ocean installations and the prediction of sediment transport and waste out-fall movements.

In this paper, a number of the statistical properties of a random wave field that have not hitherto been available in the literature are derived. Specifically, they are the probability density function and the first three statistical moments of the velocity, acceleration and dynamic pressure. The deep-water gravity waves considered are assumed to be Gaussian, of zero mean, stationary in time and homogeneous in space. Using infinitesimal wave solutions it is shown that, to first order, the velocity, acceleration and dynamic pressure are non-Gaussian. While the horizontal accelerations and vertical velocity component are of zero mean and unskewed, the vertical acceleration, horizontal velocity components and dynamic pressure are skewed and have non-zero mean.

2. Specification of random sea

Consider a rectangular co-ordinate system with the z axis vertically upwards and origin in the equilibrium surface. Let the free-surface elevation $z = \zeta(\mathbf{x}, t)$, assumed to be of zero mean and Gaussian, be represented by

$$\zeta(\mathbf{x}, t) = \int_{\mathbf{k}} dB(\mathbf{k}) \exp[i(\mathbf{k}\,.\,\mathbf{x} - nt)], \qquad (1)$$

where $\mathbf{x}$ is the horizontal position vector, t is time, $dB(\mathbf{k})$ is a complex random function of the wavenumber vector $\mathbf{k}$ and n is the frequency. Under the assumption of an inviscid, incompressible and irrotational fluid, the associated velocity potential $\phi(\mathbf{x}, z, t)$, in deep water, is given to first order by

$$\phi(\mathbf{x}, z, t) = -i \int_{\mathbf{k}} \frac{n}{|\mathbf{k}|} dB(\mathbf{k})\, e^{|\mathbf{k}|z} \exp\left[i(\mathbf{k}\,.\,\mathbf{x} - nt)\right]$$

with the frequency

$$n = (g|\mathbf{k}|)^{\frac{1}{2}}. \tag{2}$$

Denoting the unit vector parallel to the z axis by $\mathbf{e}_3$, the velocity is

$$\mathbf{u}(\mathbf{x}, z, t) = \nabla\phi(\mathbf{x}, z, t) = -i \int_{\mathbf{k}} (i\mathbf{k} - |\mathbf{k}|\,\mathbf{e}_3) \frac{n}{|\mathbf{k}|} dB(\mathbf{k})\, e^{|\mathbf{k}|z} \exp\left[i(\mathbf{k}\,.\,\mathbf{x} - nt)\right] \tag{3}$$

and the acceleration is, to first order,

$$\mathbf{a}(\mathbf{x}, z, t) = \frac{\partial \mathbf{u}(\mathbf{x}, z, t)}{\partial t} = -g \int_{\mathbf{k}} (i\mathbf{k} - |\mathbf{k}|\,\mathbf{e}_3)\, dB(\mathbf{k})\, e^{|\mathbf{k}|z} \exp\left[i(\mathbf{k}\,.\,\mathbf{x} - nt)\right] \tag{4}$$

everywhere below the free surface. Stated explicitly,

$$\bar{\mathbf{u}}(\mathbf{x}, z, t) = \mathbf{u}(\mathbf{x}, z, t)\, H(\zeta(\mathbf{x}, t) - z) \tag{5}$$

and

$$\bar{\mathbf{a}}(\mathbf{x}, z, t) = \mathbf{a}(\mathbf{x}, z, t)\, H(\zeta(\mathbf{x}, t) - z), \tag{6}$$

where $H(\;)$ is the Heaviside unit function. Similarly, the dynamic pressure is, to first order,

$$\bar{p}(\mathbf{x}, z, t) = p(\mathbf{x}, z, t)\, H(\zeta(\mathbf{x}, t) - z), \tag{7}$$

where

$$p(\mathbf{x}, z, t) = -\rho \frac{\partial \phi(\mathbf{x}, z, t)}{\partial t} = \rho g \int_{\mathbf{k}} dB(\mathbf{k})\, e^{|\mathbf{k}|z} \exp\left[i(\mathbf{k}\,.\,\mathbf{x} - nt)\right], \tag{8}$$

ρ being the water density.

It is immediately clear from (3) and (5) that, while $\mathbf{u}(\mathbf{x}, z, t)$ is Gaussian, $\bar{\mathbf{u}}(\mathbf{x}, z, t)$, being a nonlinear function of the Gaussian processes $\mathbf{u}(\mathbf{x}, z, t)$ and $\zeta(\mathbf{x}, t)$, is non-Gaussian. The same observation may also be made for $\bar{\mathbf{a}}(\mathbf{x}, z, t)$ and $\bar{p}(\mathbf{x}, z, t)$.

3. Statistical properties of velocity

The probability density functions of the velocity components $(\bar{u}_1, \bar{u}_2, \bar{u}_3)$ of $\bar{\mathbf{u}}$ are

$$f_{\bar{u}_j}(\bar{u}_j) = f_{\bar{u}_j|\zeta \leqslant z}(\bar{u}_j)\, P(\zeta \leqslant z) + f_{\bar{u}_j|\zeta > z}(\bar{u}_j)\, P(\zeta > z), \quad j = 1, 2, 3, \tag{9}$$

by the theorem of total probability. In (9), $P(\;)$ denotes the probability of the event in the parentheses while $f_{\,|}(\;)$ is the conditional probability density function.

Since ζ is Gaussian, the probability that $\zeta \leqslant z$ is

$$P(\zeta \leqslant z) = 1 - Q(z/\sigma_\zeta),$$

where

$$Q(\eta) = \int_\eta^\infty Z(\xi)\, d\xi, \quad Z(\xi) = \frac{1}{(2\pi)^{\frac{1}{2}}} \exp\left(-\tfrac{1}{2}\xi^2\right)$$

and from (1)

$$\sigma_\zeta = \left[\int_{\mathbf{k}} \Phi(\mathbf{k})\, d\mathbf{k}\right]^{\frac{1}{2}}$$

is the standard deviation of the surface elevation, whose wavenumber spectrum is $\Phi(\mathbf{k})$.

The conditional probability density function of $\overline{u}_j$, given $\zeta \leqslant z$, is

$$f_{\overline{u}_j|\zeta\leqslant z}(\overline{u}_j) = \delta(\overline{u}_j), \quad j = 1, 2, 3,$$

$\delta(\;)$ being the Dirac delta function. By the definition of conditional probability and (5),

$$f_{\overline{u}_j|\zeta>z}(\overline{u}_j) = \int_z^\infty f_{u_j\zeta}(\overline{u}_j, \zeta)\,d\zeta/P(\zeta > z), \quad j = 1, 2, 3,$$

$f_{u_j\zeta}(\overline{u}_j, \zeta)$ being the joint Gaussian probability density function of u_j and ζ.

After performing the integration and substituting into (9),

$$f_{\overline{u}_j}(\overline{u}_j) = \left[1 - Q\left(\frac{z}{\sigma_\zeta}\right)\right]\delta(\overline{u}_j) + \frac{1}{\sigma_{u_j}} Z\left(\frac{\overline{u}_j}{\sigma_{u_j}}\right) Q\left[\left(\frac{z}{\sigma_\zeta} - \frac{r_j \overline{u}_j}{\sigma_{u_j}}\right)\Big/(1-r_j^2)^{\frac{1}{2}}\right], \quad j = 1, 2, 3, \tag{10}$$

where the standard deviations of the u_j are

$$\sigma_{u_j} = \left[\int_{\mathbf{k}} \frac{|\mathbf{k}_j|^2}{|\mathbf{k}|^2} n^2 \Phi(\mathbf{k})\, e^{2|\mathbf{k}|z}\, d\mathbf{k}\right]^{\frac{1}{2}}, \quad j = 1, 2, \tag{11}$$

and

$$\sigma_{u_3} = \left[\int_{\mathbf{k}} n^2 \Phi(\mathbf{k})\, e^{2|\mathbf{k}|z}\, d\mathbf{k}\right]^{\frac{1}{2}} \tag{12}$$

from (3), and the correlation coefficients of the u_j and ζ are

$$r_j = \left[\int_{\mathbf{k}} n \frac{k_j}{|\mathbf{k}|} \Phi(\mathbf{k})\, e^{|\mathbf{k}|z}\, dk\right]\Big/\sigma_{u_j}\sigma_\zeta, \quad j = 1, 2, \quad r_3 = 0 \tag{13}$$

from (1) and (3). In the above equations $\mathbf{k}_j$ is the jth component of $\mathbf{k}$. From (10), it may be seen that, far below the equilibrium surface, the first term on the right-hand side becomes vanishingly small and the second term approaches a Gaussian form.

The first three statistical moments of $\overline{u}_j$, $j = 1, 2, 3$, can be obtained from (10) through the definition of statistical moments. They are

$$E[\overline{u}_j] = r_j \sigma_{u_j} Z\left(\frac{z}{\sigma_\zeta}\right), \quad E[\overline{u}_j^2] = \sigma_{u_j}^2\left[Q\left(\frac{z}{\sigma_\zeta}\right) + r_j^2 \frac{z}{\sigma_\zeta} Z\left(\frac{z}{\sigma_\zeta}\right)\right] \quad (14), (15)$$

and

$$E[\overline{u}_j^3] = \sigma_{u_j}^3 Z\left(\frac{z}{\sigma_\zeta}\right)\left[3r_j - r_j^3\left(1 - \frac{z^2}{\sigma_\zeta^2}\right)\right], \tag{16}$$

where $E[\;]$ denotes the expected value of the argument.

It is noted that, since $r_3 = 0$, $\overline{u}_3$ is of zero mean and unskewed though non-Gaussian. However, $\overline{u}_1$ and $\overline{u}_2$ have non-zero first-order means and are skewed except when $|z|$ is large, while for each velocity component $E[\overline{u}_j^2]$ approaches $\sigma_{u_j}^2$ with increasing depth. From (15), it may be noted that the mean kinetic energy per unit mass associated with the point under consideration is not equally divided between the horizontal and vertical motions since the point does not always remain below the water surface, as was observed by Phillips (1961).

The statistical moments of the $\overline{u}_j$ may also be derived directly without resort to explicit expressions for their probability density functions. That is, from (5),

$$E[\overline{u}_j] = E[u_j H(\zeta - z)] = E[H(\zeta - z)\, E[u_j|\zeta]]$$

(Parzen 1964), where $E[\,|\,]$ denotes 'conditional expectation'. The quantities u_j and ζ, being jointly Gaussian, yield

$$E[u_j|\zeta] = \zeta r_j \sigma_{u_j}/\sigma_\zeta,$$

giving
$$E[\overline{u}_j] = r_j \frac{\sigma_{u_j}}{\sigma_\zeta} E[\zeta H(\zeta - z)] = r_j \sigma_{u_j} Z\left(\frac{z}{\sigma_\zeta}\right)$$

as is shown in (14).

For the second statistical moment

$$E[\overline{u}_j^2] = E[u_j^2 H(\zeta - z)] = E[H(\zeta - z) E[u_j^2|\zeta]].$$

But
$$E[u_j^2|\zeta] = \sigma_{u_j}^2(1 - r_j^2) + r_j^2 \sigma_{u_j}^2 \sigma_\zeta^{-2} \zeta^2.$$

Thus
$$E[\overline{u}_j^2] = E[H(\zeta - z)]\, \sigma_{u_j}^2 (1 - r_j^2) + E[\zeta^2 H(\zeta - z)]\, r_j^2 \sigma_{u_j}^2/\sigma_\zeta^2.$$

Noting that
$$E[H(\zeta - z)] = Q(z/\sigma_\zeta)$$

and
$$E[\zeta^2 H(\zeta - z)] = \sigma_\zeta^2 \left[\frac{z}{\sigma_\zeta} Z\left(\frac{z}{\sigma_\zeta}\right) + Q\left(\frac{z}{\sigma_\zeta}\right)\right],$$

(15) is recovered.

The quantity $E[\overline{u}_j^3]$ in (16) may be arrived at in much the same manner as $E[\overline{u}_j]$ and $E[\overline{u}_j^2]$, and is therefore not rederived here.

4. Statistical properties of acceleration and pressure

Examination of (5)–(7) suggests that the probability density functions and statistical moments of $\overline{\mathbf{a}}(\mathbf{x}, z, t)$ and $\overline{p}(\mathbf{x}, z, t)$ are of the same form as those of $\overline{\mathbf{u}}(\mathbf{x}, z, t)$ as given by (10) and (14)–(16).

Thus, for $\overline{a}_j(\mathbf{x}, z, t)$, $\overline{u}_j$ in (10) and (14)–(16) is to be replaced by $\overline{a}_j$, and σ_{u_j} by the standard deviation of a_j:

$$\sigma_{a_j} = g\left[\int_{\mathbf{k}} |\mathbf{k}_j^2|\, \Phi(\mathbf{k})\, e^{2|\mathbf{k}|z}\, d\mathbf{k}\right]^{\frac{1}{2}}, \quad j = 1, 2,$$

$$\sigma_{a_3} = g\left[\int_{\mathbf{k}} |\mathbf{k}|^2\, \Phi(\mathbf{k})\, e^{2|\mathbf{k}|z}\, d\mathbf{k}\right]^{\frac{1}{2}}.$$

Also,
$$r_1 = r_2 = 0, \quad r_3 = g\left[\int_{\mathbf{k}} |\mathbf{k}|\, \Phi(\mathbf{k})\, e^{|\mathbf{k}|z}\, d\mathbf{k}\right] \Big/ \sigma_{a_3}\sigma_\zeta.$$

The horizontal components of the acceleration are, therefore, of zero mean and unskewed.

Similarly, for the dynamic pressure $\overline{p}(\mathbf{x}, z, t)$, $\overline{p}$ is to be used in place of $\overline{u}_j$ in (10) and (14)–(16), and σ_{u_j} should be replaced by

$$\sigma_p = \rho g\left[\int_{\mathbf{k}} \Phi(\mathbf{k})\, e^{2|\mathbf{k}|z}\, d\mathbf{k}\right]^{\frac{1}{2}}.$$

In those equations r_j is now

$$r = \rho g\left[\int_{\mathbf{k}} \Phi(\mathbf{k})\, e^{|\mathbf{k}|z}\, d\mathbf{k}\right] \Big/ \sigma_p \sigma_\zeta.$$

It should be mentioned here that at $z = 0$ $r = 1$, in which case the probability density function is

$$f_{\bar{p}}(\bar{p}) = \tfrac{1}{2}\delta(\bar{p}) + \frac{1}{\sigma_p} Z\left(\frac{\bar{p}}{\sigma_\zeta}\right) H(\bar{p})$$

while the statistical moments still retain the forms (14)–(16).

This work was supported in part by the Center for Coastal Marine Studies of North Carolina State University.

REFERENCES

PARZEN, E. 1964 *Stochastic Processes*. Holden-Day.

PHILLIPS, O. M. 1961 On the dynamics of unsteady gravity waves of finite amplitude. Part 2. *J. Fluid Mech.* **11**, 143–155.